Double Integrals (Part II)

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1 Double Integral on General Regions

In many cases, we want to compute the volume of some solid whose base is not a rectangle. In such situation, we need to define and compute the double integrals over a general region. The idea is to extend the given function to a larger rectangular domain by assigning the value 0 to the points out of D:

We define $\iint_D f(x,y)\,dA = \iint_R \tilde{f}(x,y)\,dA$, where R is a rectangular region enclosing D and

$$\tilde{f}(x,y) = \begin{cases} f(x,y), \text{if } (x,y) \text{ is in } D \\ 0, \text{if } (x,y) \text{ is not in } D \end{cases}$$

Proposition 1. *D is a region on the xy-plane, then:*

1. $\iint_D f(x,y) + g(x,y)\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$
2. $\iint_D cf(x,y)\,dA = c\iint_D f(x,y)\,dA$
3. *If $f(x,y) \geq g(x,y)$ on D, then $\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$*

A region D is said to be of Type I if it is bounded by $x = a$, $x = b$, $y = g_1(x)$ and $y = g_2(x)$, where g_1, g_2 are continuous functions in x and $g_1(x) \leq g_2(x)$ on $[a, b]$.

The double integral over a Type I region can be computed as follows:

Theorem 2.

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

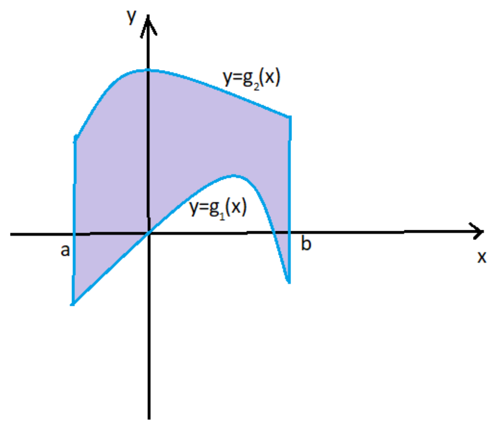


Figure 1: Type I region

Proof. We use a rectangle $R = [a, b] \times [c, d]$ to enclose this region D.

$$
\begin{aligned}
\iint_D f(x, y)\, dA &= \iint_R \tilde{f}(x, y)\, dA \\
&= \int_a^b \int_c^d \tilde{f}(x, y)\, dy\, dx \\
&= \int_a^b \Big(\int_c^{g_1(x)} \tilde{f}(x, y)\, dy + \int_{g_1(x)}^{g_2(x)} \tilde{f}(x, y)\, dy + \int_{g_2(x)}^{d} \tilde{f}(x, y)\, dy \Big)\, dx \\
&= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx
\end{aligned}
$$

□

Similarly, we can define a plane region to be of Type II if it it bounded by $y = c$,$y = d$, $x = h_1(y)$ and $x = h_2(y)$, where h_1, h_2 are continuous functions on $[c, d]$ and $h_1(y) \le h_2(y)$ on $[c, d]$.

Then the integral on D is given by

Theorem 3.

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

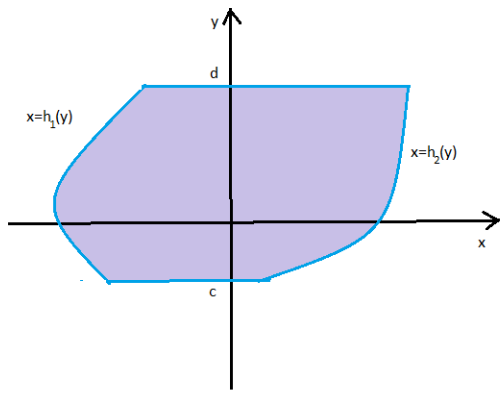
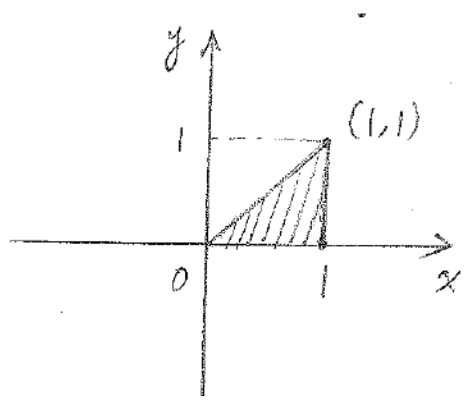


Figure 2: Type II region

Example 4. *Compute the integral* $\iint_D xy\,dA$ *over the triangular shaded region shown in the following figure.*



Method I:

$$\iint_D f(x,y)\,dA = \int_0^1 \int_0^x xy\,dy\,dx = \int_0^1 (\frac{x}{2}y^2\Big|_0^x)\,dx$$

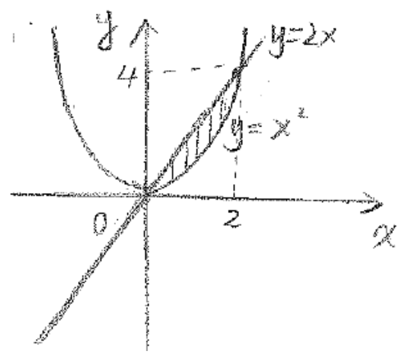
$$= \int_0^1 \frac{x^3}{2}\,dx$$

$$= \frac{1}{8}$$

Method II:

$$\iint_D f(x,y)\,dA = \int_0^1 \int_y^1 xy\,dx\,dy$$
$$= \int_0^1 \left(\frac{y}{2}x^2\Big|_y^1\right) dy$$
$$= \int_0^1 \frac{y-y^3}{2}\,dy$$
$$= \frac{1}{8}$$

Example 5. *Find the volume of the solid that lies under the graph $z = f(x,y) = x^2 + y^2$ and above the region D in xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.*



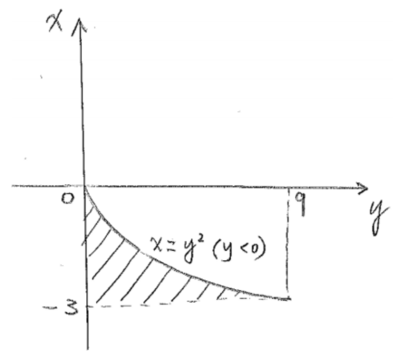
$$\iint_D f(x,y)\,dA = \int_0^2 \int_{x^2}^{2x} x^2 + y^2\,dy\,dx$$
$$= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14}{3}x^3\right) dx$$
$$= \frac{216}{35}$$

Example 6. *Rewrite the integral in the above example in the form of $\iint f(x,y)\,dx\,dy$*

$$\iint_D f(x,y)\,dA = \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} x^2 + y^2\,dx\,dy$$

Example 7. *Rewrite the iterated integral* $\int_{-3}^{0} \int_{0}^{y^2} f(x,y)\,dx\,dy$ *in the form of* $\int \int f(x,y)\,dy\,dx$

By the given iterated integral, we can recover the region D *to be the following:*



So the integral can be written as

$$\int_0^9 \int_{-3}^{-\sqrt{x}} f(x,y)\,dy\,dx$$

Proposition 8. *Given a region* D *on xy-plane, its area is*

$$\iint_D 1\,dA$$

Example 9. *Compute the area bounded between the curve* $x = y^2$ *and* $x = 4$

$$\iint_D 1\,dA = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} 1\,dy\,dx = \int_0^4 2\sqrt{x}\,dx = \frac{32}{3}$$

An important application of double integral is to compute the mass of some thin object.

Proposition 10. *If a thin object is put on the xy-plane, it occupies a region* D*. If the density function of this object is* $\rho(x,y)$*, then its mass is given by*

$$\iint_D \rho(x,y)\,dA$$